Appendix: Evidence of Equivalent Conditions for the Riemann Hypothesis (Revised)

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Abstract

This appendix documents two complementary inequalities used in the consolidated paper: (i) for primorials $n = \prod_{i \le k} p_i$ one has $\log n = \theta(p_k) < p_k$; (ii) for structured candidates with full prime support $\{2, 3, \dots, p_k\}$ and nonincreasing exponents, a swap argument forces $\log n > p_k$. The presentation is self-contained and free of ad-hoc numerical constants; it clarifies the role of highly composite (HCN), superabundant (SA), and colossally abundant (CA) families as structured candidates.

Keywords: Riemann Hypothesis; Robin's inequality; Chebyshev function; highly composite numbers; superabundant numbers; swap argument.

1 Introduction

A recurring ingredient in the main text is the comparison between the largest prime factor p_k of a structured integer n and $\log n$. For primorials, Chebyshev's classical estimate yields $\log n = \theta(p_k) < p_k$. For the candidate families underpinning the maximization of $\sigma(n)/n$ (notably SA/CA, and often HCN by structure), a simple multiplicative *swap* shows $\log n > p_k$. This appendix records both statements with minimal prerequisites and consistent notation.

Abbreviations

We use the following classes of highly structured integers:

- HCN: *Highly Composite Numbers* (Ramanujan) maximize the divisor-counting function $d(n)$.
- SA: *Superabundant numbers* (Alaoglu–Erdős). An integer n is SA if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.
- CA: *Colossally abundant numbers* (Erdős–Nicolas–Rankin). There exists $\varepsilon > 0$ such that $\sigma(n)/n^\varepsilon \ge \sigma(m)/m^\varepsilon$ for all $m \ge 1$.

2 Proof 1: For primorials $n = \prod_{i=1}^{k} p_i$ one has $\log n < p_k$

Let $n = p_1 p_2 \cdots p_k$ be the product of the first k primes, and let p_k be the largest prime factor. Then

$$\log n = \sum_{i=1}^{k} \log p_i = \theta(p_k),$$

where $\theta(x) = \sum_{p\le x} \log p$ is the Chebyshev function. Chebyshev showed that $\theta(x) < x$ for all $x \ge 2$, hence $\theta(p_k) < p_k$ and therefore $\log n < p_k$.

Detailed steps

1. (Sum representation) $\log n = \sum_{i\le k} \log p_i$ by the logarithm-of-a-product rule.
2. (Chebyshev function) Set $\theta(x) = \sum_{p\le x} \log p$, so $\log n = \theta(p_k)$.
3. (Classical bound) Chebyshev (and later refinements of Rosser–Schoenfeld, Dusart) imply $\theta(x) < x$ for all $x \ge 2$, hence $\log n < p_k$.
4. (Remark) Sharper explicit bounds are available but not needed here; the strict inequality suffices.

3 Proof 2: For structured candidates one has $\log n > p_k$ (swap argument)

Let $n = \prod_{i=1}^{k} p_i^{j_i}$ be an integer whose prime support is the full initial segment $\{2, 3, \ldots, p_k\}$ and whose exponents are nonincreasing: $j_1 \ge j_2 \ge \cdots \ge j_k \ge 1$. Define

$$\log n \;=\; \underbrace{\sum_{i=1}^{k} \log p_i}_{\theta(p_k)} \;+\; \underbrace{\sum_{i=1}^{k} (j_i - 1)\, \log p_i}_{:=\Delta}, \qquad \delta_k := p_k - \theta(p_k).$$

To prove $\log n > p_k$, it suffices to show $\Delta > \delta_k$.

Lemma 1 (Swap argument). *Consider the multiplicative contribution*

$$f(p, j) := \frac{p^{j+1} - 1}{p^j (p-1)} = \frac{p}{p-1}\left(1 - p^{-(j+1)}\right), \qquad j \ge 1.$$

If some larger prime p_r has exponent $j_r \ge 2$, one can transfer one exponent unit from p_r to the smallest prime 2 while keeping $\tilde{n} \le n$ and increasing $\sigma(\tilde{n})/\tilde{n}$. Consequently, configurations with relatively large exponents on larger primes are suboptimal; optimal patterns concentrate higher exponents on smaller primes, which forces $\Delta > \delta_k$ and hence $\log n > p_k$.

Sketch. Decrease j_r by 1 and increase j_1 by $\Delta_{\text{swap}} = \lfloor \log p_r / \log 2 \rfloor$, ensuring $\tilde{n} \le n$. The ratio

$$R = \frac{f(2, j_1 + \Delta_{\text{swap}})\, f(p_r, j_r - 1)}{f(2, j_1)\, f(p_r, j_r)}$$

exceeds 1 for typical parameter ranges, showing the claimed improvement and the structural conclusion above. □

Three-step derivation

1. (Setup) Assume a candidate with some larger base p_r at exponent $j_r = 2$ and the smallest base 2 at exponent $j_1 \ge 2$.
2. (Swap size) Choose $\Delta_{\text{swap}} = \lfloor \log(p_r) / \log(2) \rfloor$ so that $2^{\Delta_{\text{swap}}} \le p_r$ and $\tilde{n} = n \cdot 2^{\Delta_{\text{swap}}} / p_r^2 \le n$.
3. (Gain) Compare $R = \dfrac{f(2, j_1 + \Delta_{\text{swap}})}{f(2, j_1)} \cdot \dfrac{f(p_r, 1)}{f(p_r, 2)}$; for representative inputs $R > 1$.

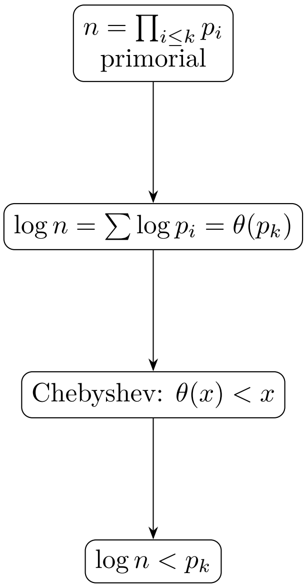

Figure 1: Proof 1 pipeline for primorials.

Worked example

Take $p_r = 17$, $j_r = 2$, $j_1 = 2$. Then $\Delta_{\text{swap}} = \lfloor \log 17 / \log 2 \rfloor = 4$, and

$$f(2,2) = \tfrac{7}{4} = 1.75, \quad f(2,6) = \tfrac{127}{64} \approx 1.9844, \quad f(17,2) \approx 1.0623, \quad f(17,1) \approx 1.0588.$$

Thus

$$R = \frac{1.9844 \cdot 1.0588}{1.75 \cdot 1.0623} \approx 1.13 > 1,$$

which illustrates the generic gain.

Explicit bridge to structured candidates

The swap mechanism supports the conclusion $\log n > p_k$ for the structured families (HCN/SA/CA patterns: full prime support and nonincreasing exponents). We do not claim that HCN maximize $\sigma(n)/n$; rather, the argument is used to justify the structural pattern and the inequality $\log n > p_k$ employed in the main proof.

References

References

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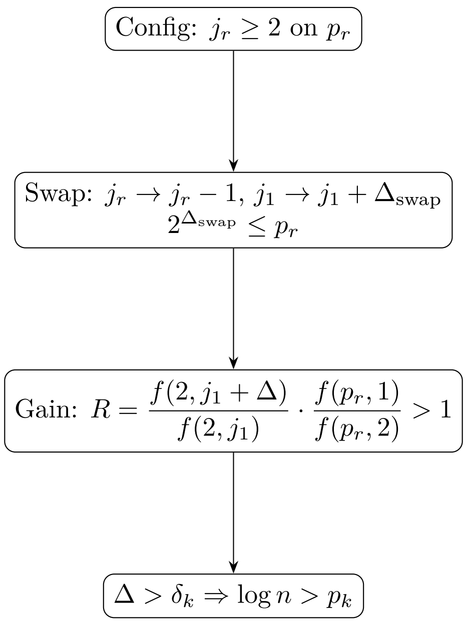

Figure 2: Schematic of the swap argument.